Concavity and the Second Derivative Test

The graph of a differentiable function $y = f(x)$ is:

a) **Concave up** on an open interval I if y' is increasing on I.

b) **Concave down** on an open interval I if y' is decreasing on I.

-If a function has a second derivative, then we can conclude that y' increases if $y'' > 0$ and y' decreases if $y'' < 0$.

Concavity Test

-The graph of a twice differentiable function $y = f(x)$ is

a) **Concave up** on any interval where $y'' > 0$

b) **Concave down** on any interval where $y'' < 0$

Example

-Find the concavity of $y = 3 + \sin\left(x\right)$ on $\left[0, 2\pi\right]$

$$f'\left(x\right) = \cos\left(x\right)$$

$$f''\left(x\right) = -\sin\left(x\right)$$

-Concave down on $\left(0, \pi\right)$ where $-\sin\left(x\right)$ is negative.

-Concave up on $\left(\pi, 2\pi\right)$ where $-\sin\left(x\right)$ is positive.

Points of Inflection

-The curve $y = 3 + \sin\left(x\right)$ changes concavity at the point $\left(\pi, 3\right)$.

-We call $\left(\pi, 3\right)$ a **point of inflection**.

-A point where the graph of a function has a tangent line and where the concavity changes is a **point of inflection**.

-At such a point $y'' = 0$ or is undefined.

-If $y'' = 0$ at a point of inflection then y' has a local min or max.

Example-Motion Along a Line

$$s\left(t\right) = 2t^3 - 14t^2 + 22t - 5 \qquad t \geq 0$$

-Find the velocity, acceleration, and describe the motion of the particle.

$$v\left(t\right) = s'\left(t\right) = 6t^2 - 28t + 22 = 2\left(t-1\right)\left(3t-11\right)$$

$$a(t) = v'(t) = s''(t) = 12t - 28 = 4(3t - 7)$$

-When $s(t)$ is increasing the particle is moving right. Left when $s(t)$ is decreasing.

-Notice that the 1st derivative $(v = s')$ is 0 when $t = 1$ and $t = 11/3$.

Intervals	$0 < t < 1$	$1 < t < 11/3$	$11/3 < t$
Sign of v	+	-	+
Behavior of s	Increasing	Decreasing	Increasing
Motion	Right	Left	Right

-Moving to the right on $[0,1)$ and $(11/3, \infty)$ and moving to the left on $(1, 11/3)$.

-$a(t) = v'(t) = s''(t) = 12t - 28 = 4(3t - 7)$ is zero when $t = 7/3$

Intervals	$0 < t < 7/3$	$7/3 < t$
Sign of a	-	+
Graph of s	Concave down	Concave up

-The acceleration force is directed left during $[0, 7/3)$, is momentarily 0 at $t = 7/3$ and directed right afterwards.

Second Derivative Test for LOCAL EXTREMA

1) If $f'(c) = 0$ and $f'' < 0$, then f has a **local maximum** at $x = c$.

2) If $f'(c) = 0$ and $f'' > 0$, then f has a **local minimum** at $x = c$.

-The test fails if $f''(c) = 0$ or $f''(c)$ DNE.

-When this happens find local extremas from the first derivative.

Example-Using the 2nd Derivative Test

$f(x) = x^3 - 12x - 5$

$f'(x) = 3x^2 - 12 = 3(x^2 - 4)$

$f''(x) = 6$

-Test critical points $x = \pm 2$ (there are no endpoints)

$f''(-2) = -12 < 0$ ➔ f has a local max at $x = -2$

$f''(2) = 12 > 0$ ➔ f has a local min at $x = 2$

Example-Using f' and f'' to graph f

$f'(x) = 4x^3 - 12x^2 = 4x^2(x - 3)$

-The first derivative is zero at $x = 0$ and $x = 3$

Intervals	$x < 0$	$0 < x < 3$	$3 < x$
Sign of f'	-	-	+
Behavior of f	Decrease	Decrease	Increase

-No extrema at $x = 0$ and a local min at $x = 3$.

-f is decreasing on $(-\infty, 0]$ and $[0, 3]$ and increasing in $[3, \infty)$.

-$f''(x) = 12x^2 - 24x = 12x(x - 2)$ is zero at $x = 0$ and $x = 2$

Intervals	$x < 0$	$0 < x < 2$	$2 < x$

Sign of f''	+	-	+
Behavior or f	Concave up	Concave down	Concave up

-f is concave up on $(-\infty, 0)$ and $(2, \infty)$ and down on $(0, 2)$

-combining tables we get

$x < 0$	$0 < x < 2$	$2 < x < 3$	$x > 3$
Decreasing	Decreasing	Decreasing	Increasing
Concave up	Concave down	Concave up	Concave up